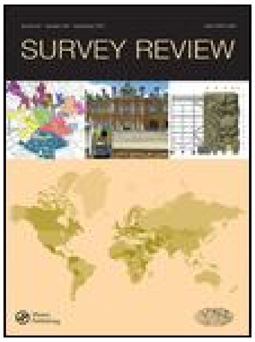

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On the errors-in-variables model with inequality constraints of dependent variables for geodetic transformation

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The Total least-squares (TLS) adjustment with inequality constraints has received increased attention in geodesy over the last three years. In the most recent work, inequality constraints have been presented that can restrict unknown parameters and independent variables, but no one has provided an inequality-constrained adjustment for restricting dependent variables. In this work, we review the TLS adjustment methods in terms of different model formulations and then investigate the errors-in-variables model with inequality constraints for dependent variables. Finally, we demonstrate the practicality of our approach with a planar geodetic transformation, where the uncertainty of the target observations is reduced via the inequality constraints for dependent variables.

Keywords: Total least-squares, Errors-in-variables model, Inequality constraints, Dependent variables, Geodetic transformation, Uncertainty reduction

Introduction

In recent years, there has been a renewed interest in geodetic datum conversion methods due to the need to transform data measured in old coordinate systems to high-precision GNSS-based data (Felus and Burtch 2009). In these cases, the total least-squares (TLS) adjustment method is frequently used to reach the desired accuracy. Actually, it has been known for more than a century that the errors-in-variables (EIV) model can be treated as a special case of the nonlinear Gauss–Helmert model (GHM). This implies that the EIV model can be adjusted by the least-squares (LS) method in an iteratively linearised model, as shown by Helmert, Deming (1931, 1934) and, later, in Neitzel (2010).

Since the seminal paper by Golub and Van Loan (1980), investigation has been attempted in various ways. Regarding the limitation of the stochastic model (covariance matrix of all random errors), the original TLS approach, which could tolerate only diagonal covariance matrices, has been generalised in several steps, by Schaffrin and Wieser (2008), Fang (2011), Amiri-Simkooei and Jazaeri (2012), Mahboub (2012), Snow (2012), Xu *et al.* (2012), Amiri-Simkooei (2013) and Fang (2013, 2014a, 2015), to accept any non-negative covariance matrices. Further progress has been made towards the direct use of existing adjustment models, which can be used to reformulate or reinterpret the functional part of the EIV model.

In contrast to the iteratively linearised GHM, nonlinear TLS algorithms have been developed in different ways. Schaffrin and Wieser (2011) transformed the standard EIV model into the condition adjustment model. In this model transformation, a coefficient matrix eliminates the corrected design matrix, where the sum of the ranks of both matrices equals the length of the observation vector. Teunissen (1988) reformulated the EIV model as an extended Gauss Markov adjustment model, in which the elements in the design matrix are regarded as unknowns (also see the generalisation of Xu *et al.* 2012). Schaffrin (2013) showed that the EIV model can be interpreted as a set of direct observation equations with nonlinear constraints, which were proved to be equivalent to the orthogonal regression applied by Deming (1931, 1934). Furthermore, Amiri-Simkooei and Jazaeri (2012) and Jazaeri *et al.* (2014) formulated the EIV model using the standard least-squares theory.

The EIV model with inequality constraints is another important extension that has been investigated recently. Zhang *et al.* (2013) proposed a combinatorial strategy to adjust the EIV model with inequality constraints, whereas Fang (2014b) and Fang and Wu (2015) used the non-combinatorial strategy to obtain the inequality-constrained TLS (ICTLS) solution. Later, Zeng *et al.* (2015) proposed an iterative ICTLS algorithm based on iteratively solving a linear complimentary problem. Their algorithm can also account for restrictions on the independent variables (the elements within the design matrix). However, until now, it has not been clear how to restrict dependent variables (elements within the observation vector) by inequalities in the symmetrical adjustment. Therefore, it is theoretically meaningful that the

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independent variables as well as the dependent variables can be restricted simultaneously in the symmetrical adjustment. Note that in mathematical modelling, dependent variables are studied to see how much they vary with the independent variables. Practically, the prior information – inequality constraints – can be obtained from the previous transformation adjustment, which restricts the estimated observations or the predicted residuals in a trusted interval in the full analogy of reduction of measurement uncertainty.

In this work, we first review the adjustment strategies of the EIV model based on different conventional models. Then an ICTLS solution is proposed based on direct observation with constraints, which can restrict both independent and dependent variables. Finally, we show the application of the geodetic transformation and discuss our conclusions.

Reformulation of EIV model to conventional adjustment models

Let the standard EIV model be defined by the functional and stochastic model

$$\mathbf{y} + \mathbf{v_y} = (\mathbf{A} + \mathbf{V_A})\boldsymbol{\xi} \tag{1}$$

$$\mathbf{v} := \begin{bmatrix} vec(\mathbf{V_A}) \\ \mathbf{v_y} \end{bmatrix} = \begin{bmatrix} \mathbf{v_A} \\ \mathbf{v_y} \end{bmatrix} \sim \left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \sigma_0^2 \mathbf{Q} \right) \tag{2}$$

where

- $\mathbf{y}$ and $\mathbf{v_y}$ are the observation and random correction vectors, respectively;
- $\mathbf{A}$ and $\mathbf{V_A}$ are the full-column rank stochastic coefficient matrix ($n \times m$) and the corresponding random correction matrix, respectively; $\boldsymbol{\xi}$ is the unknown parameter vector with dimension $m \times 1$;
- $\mathbf{v}$ is the extended random correction given by $\mathbf{v_A} = vec(\mathbf{V_A})$;
- σ_0^2 is the unknown/known variance factor; and
- $\mathbf{Q}$ is the non-negative definite cofactor matrix of the vector $\mathbf{v}$.

Models equivalent to the EIV model have been applied since at least Pearson (1901). In this section, we aim to reformulate the EIV model using different conventional adjustment models, namely the linearised GHM (condition equations with unknowns), condition equations, Gauss–Markov model (observation equations) and direct observation equations with nonlinear constraints.

Gauss–Helmert model

For a long time, the nonlinear GHM has been used to compute the LS solution of nonlinear models. The functional EIV model

$$\mathbf{f}(\mathbf{l} + \mathbf{v}, \boldsymbol{\xi}) = (\mathbf{A} + \mathbf{V_A})\boldsymbol{\xi} - \mathbf{y} - \mathbf{v_y} = \mathbf{0} \tag{3}$$

can be linearised through the truncated Taylor series to form the GHM (Fang and Wu 2015)

$$\mathbf{A}^i d\boldsymbol{\xi} + \mathbf{B}^i \mathbf{v} + \mathbf{w}^i = \mathbf{0} \tag{4}$$

with deterministic Jacobian matrices $\mathbf{A}^i = \mathbf{A} + \mathbf{V}_\mathbf{A}^i$ and $\mathbf{B}^i = [(\boldsymbol{\xi}^i)^T \otimes \mathbf{I}_n, -\mathbf{I}_n]$, inconsistency vector $\mathbf{w}^i$ and parameter increment vector $d\boldsymbol{\xi}$. Note that the model matrices $\mathbf{A}^i$ and $\mathbf{B}^i$ are nonrandom. In conjunction with iterative linearisation, the adjustment of the GHM can generate the TLS solution of the EIV model (Neitzel 2010).

Condition equation model

The Gauss–Markov model can be transformed into condition equations when the transposed coefficient matrix $\mathbf{S}^T$ multiplies the deterministic coefficient matrix $\mathbf{A}$ equal a zero matrix and the rank condition $rank(\mathbf{S}) + rank(\mathbf{A}) = n$ is fulfilled. A mixed formulation of an EIV model was presented in Amiri-Simkooei *et al.* (2016a). In analogy with the transformation between the Gauss–Markov model and the condition equations, Schaffrin and Wieser (2011) established the condition equations for the EIV model

$$(\mathbf{S} + \mathbf{V_S})^T(\mathbf{y} + \mathbf{v_y}) = \mathbf{0} \tag{5}$$

by using the null space condition and the rank condition

$$\begin{aligned} &(\mathbf{S} + \mathbf{V_S})^T(\mathbf{A} + \mathbf{V_A}) = \mathbf{0} \\ &rank(\mathbf{S} + \mathbf{V_S}) + rank(\mathbf{A} + \mathbf{V_A}) = n. \end{aligned} \tag{6}$$

By iteratively computing the Lagrange multipliers, the corrections $\mathbf{V_S}$ and $\mathbf{v_y}$ can be obtained after convergence.

Extended Gauss–Markov model

Teunissen (1985) was the first to point out that the TLS or EIV problem can be formulated as a simple nonlinear (bilinear) Gauss–Markov model (also see Teunissen 1988). Recently, Xu *et al.* (2012) proposed a partial EIV model to generate the TLS solution within the EIV model even when the elements in the design matrix are structured. This partial EIV model is applied in conjunction with increment of number of unknowns to simultaneously compute the original unknown parameters and the predicted values of the elements within the design matrix. Therefore, the EIV model can be reformulated as a nonlinear extended Gauss–Markov model

$$\begin{bmatrix} \mathbf{y} \\ vec(\mathbf{A}) \end{bmatrix} + \begin{bmatrix} \mathbf{v_y} \\ vec(\mathbf{V_A}) \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}\xi \\ vec(\bar{\mathbf{A}}) \end{bmatrix} \tag{7}$$

where the matrix $\bar{\mathbf{A}}$ denotes the true design matrix.

For the nonlinear Gauss–Markov model, there are various ways to compute the parameter vector $\boldsymbol{\xi}$ and the true design matrix $\bar{\mathbf{A}}$ (see Lenzmann and Lenzmann 2007 and Xu *et al.* 2012). Since the independent variables are regarded as new unknown parameters, the inequality constraints of the independent variables can be taken into account (see Zeng *et al.* 2015).

Direct observation model with nonlinear constraints

In last reformulation, the true observation vector $\bar{\mathbf{y}}$ is introduced as an additional unknown parameter vector, allowing the EIV model to be reformulated as a direct observation model with nonlinear constraints

$$\begin{aligned} &\begin{bmatrix} \mathbf{y} \\ vec(\mathbf{A}) \end{bmatrix} + \begin{bmatrix} \mathbf{v_y} \\ vec(\mathbf{V_A}) \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{y}} \\ vec(\bar{\mathbf{A}}) \end{bmatrix} \\ &\textit{subject to} \\ &\bar{\mathbf{y}} - \bar{\mathbf{A}}\boldsymbol{\xi} = \mathbf{0} \end{aligned} \tag{8}$$

where all the vectors $\bar{\mathbf{y}}$, $\bar{\mathbf{A}}$, $\boldsymbol{\xi}$ are regarded as unknowns. Since the dependent variables are regarded as the unknown parameter in this model, we might obtain the chance to consider the inequality constraints for them.

EIV model with inequality constraints for dependent variables

Although TLS estimates made using any of the equivalent formulations described above could still be identical, the last reformulation might have benefits when applied to inequality-constrained problems. For example, in the third formulation (Equation (7)), one can restrict the independent variables and the unknown parameters; however, the dependent variables cannot be restricted by incorporating linear inequality constraints since they are not regarded as unknown parameters. In contrast, the fourth formulation (Equation (8)) can explicitly handle the situation in which the inequality constraints of the dependent variables are incorporated into the model.

If one incorporates the desired linear inequalities into Equation (8), the functional part of the constrained EIV model can be expressed as follows

$$\begin{bmatrix} \mathbf{y} \\ vec(\mathbf{A}) \end{bmatrix} + \begin{bmatrix} \mathbf{v_y} \\ vec(\mathbf{V_A}) \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{y}} \\ vec(\bar{\mathbf{A}}) \end{bmatrix}$$
$$\textit{subject to} \qquad (9)$$
$$\bar{\mathbf{y}} - \bar{\mathbf{A}}\boldsymbol{\xi} = \mathbf{0}$$
$$\mathbf{C}\boldsymbol{\xi}_t \geq \mathbf{c}$$

where $\mathbf{C}$ is the fixed coefficient matrix of inequality constraints, $\mathbf{c}$ is a constant vector on the right-hand side of the inequality constraints and the extended parameter vector $\boldsymbol{\xi}_t$ denotes all unknowns including $\bar{\mathbf{A}}$, $\bar{\mathbf{y}}$ and $\boldsymbol{\xi}$. In this case, the vector of dependent variables $\bar{\mathbf{y}}$ can be fully restricted by inequality constraints.

Based on Equation (9), the objective function, in conjunction with a linearisation, can be expressed as a standard quadratic programme

$$\left(\begin{bmatrix} d\bar{\mathbf{y}} \\ vec(d\bar{\mathbf{A}}) \end{bmatrix} - \begin{bmatrix} \mathbf{y} - \bar{\mathbf{y}}^{(0)} \\ vec(\mathbf{A}) - vec(\bar{\mathbf{A}}^{(0)}) \end{bmatrix}\right)^T$$
$$\mathbf{Q}^{-}\left(\begin{bmatrix} d\bar{\mathbf{y}} \\ vec(d\bar{\mathbf{A}}) \end{bmatrix} - \begin{bmatrix} \mathbf{y} - \bar{\mathbf{y}}^{(0)} \\ vec(\mathbf{A}) - vec(\bar{\mathbf{A}}^{(0)}) \end{bmatrix}\right)$$
$$\textit{subject to}$$
$$\bar{\mathbf{y}}^{(0)} - \bar{\mathbf{A}}^{(0)}\boldsymbol{\xi}^{(0)} + \begin{bmatrix} \mathbf{I} & -\boldsymbol{\xi}^{(0)} \otimes \mathbf{I} & -\bar{\mathbf{A}}^{(0)} \end{bmatrix} \begin{bmatrix} d\bar{\mathbf{y}} \\ vec(d\bar{\mathbf{A}}) \\ d\boldsymbol{\xi} \end{bmatrix} = \mathbf{0}$$
$$\mathbf{C}\begin{bmatrix} d\bar{\mathbf{y}} \\ vec(d\bar{\mathbf{A}}) \\ d\boldsymbol{\xi} \end{bmatrix} + \mathbf{C}\begin{bmatrix} \bar{\mathbf{y}}^{(0)} \\ vec(\bar{\mathbf{A}}^{(0)}) \\ \boldsymbol{\xi}^{(0)} \end{bmatrix} \geq \mathbf{c} \qquad (10)$$

where the vectors with the index $^{(0)}$ are approximate values, and the vectors prefixed with d, denoting that they are increments, are the new unknown parameters to be determined in the quadratic form. Quadratic programming is a well-known process, and has been described in many text books (e.g. Nocedal and Wright 2006, p. 490), also see Fang (2014b) for the weighted TLS environment. Note that the objective function is invariant when the rank of the matrix $[\mathbf{ABQ}]$ equal n (see Neitzel and Schaffrin 2016). When the increment of the extended parameter vector is obtained, the next iteration starts until the given tolerance is reached. Therefore, the ICTLS algorithm for dependent variables can be briefly described as follows

Step 1) Give approximated values $\begin{bmatrix} \bar{\mathbf{y}}^{(0)} \\ vec(\bar{\mathbf{A}}^{(0)}) \\ \boldsymbol{\xi}^{(0)} \end{bmatrix} = \begin{bmatrix} \mathbf{y} \\ vec(\mathbf{A}) \\ (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y} \end{bmatrix}$;

Step 2) Implement the quadratic programme

$$\left(\begin{bmatrix} d\bar{\mathbf{y}} \\ vec(d\bar{\mathbf{A}}) \end{bmatrix} - \begin{bmatrix} \mathbf{y} - \bar{\mathbf{y}}^{(0)} \\ vec(\mathbf{A}) - vec(\bar{\mathbf{A}}^{(0)}) \end{bmatrix}\right)^T$$
$$\mathbf{Q}^{-}\left(\begin{bmatrix} d\bar{\mathbf{y}} \\ vec(d\bar{\mathbf{A}}) \end{bmatrix} - \begin{bmatrix} \mathbf{y} - \bar{\mathbf{y}}^{(0)} \\ vec(\mathbf{A}) - vec(\bar{\mathbf{A}}^{(0)}) \end{bmatrix}\right)$$
$$\textit{subject to}$$
$$\bar{\mathbf{y}}^{(0)} - \bar{\mathbf{A}}^{(0)}\boldsymbol{\xi}^{(0)} + \begin{bmatrix} \mathbf{I} & -\boldsymbol{\xi}^{(0)} \otimes \mathbf{I} & -\bar{\mathbf{A}}^{(0)} \end{bmatrix} \begin{bmatrix} d\bar{\mathbf{y}} \\ vec(d\bar{\mathbf{A}}) \\ d\boldsymbol{\xi} \end{bmatrix} = \mathbf{0}$$
$$\mathbf{C}\begin{bmatrix} d\bar{\mathbf{y}} \\ vec(d\bar{\mathbf{A}}) \\ d\boldsymbol{\xi} \end{bmatrix} + \mathbf{C}\begin{bmatrix} \bar{\mathbf{y}}^{(0)} \\ vec(\bar{\mathbf{A}}^{(0)}) \\ \boldsymbol{\xi}^{(0)} \end{bmatrix} \geq \mathbf{c}$$

to obtain the increments $\begin{bmatrix} d\hat{\mathbf{y}} \\ vec(d\hat{\mathbf{A}}) \\ d\hat{\boldsymbol{\xi}} \end{bmatrix}$.

Step 3) Create the new approximate values using

$$\begin{bmatrix} \bar{\mathbf{y}}^{(1)} \\ vec(\bar{\mathbf{A}}^{(1)}) \\ \boldsymbol{\xi}^{(1)} \end{bmatrix} = \begin{bmatrix} d\hat{\mathbf{y}} \\ vec(d\hat{\mathbf{A}}) \\ d\hat{\boldsymbol{\xi}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{y}}^{(0)} \\ vec(\bar{\mathbf{A}}^{(0)}) \\ \boldsymbol{\xi}^{(0)} \end{bmatrix}$$

Then repeat Steps 2 and 3 until the norm of the increments is smaller than 10^{-10}.

When the iterative process is terminated, the variance factor can be approximated by the values of the objective function (the total squared sum of residuals, TSSR) divided by the degrees of freedom $(n - m + c_a)$, where c_a is the number of active inequality constraints. Note that the mean squared errors for the parameter vector need to be investigated in future.

Since the bias analysis is difficult for the inequality-constrained LS problem, one could use the bias detection technique after the constraints are treated as pseudo-observation equations. Regarding the convergence of the iteration, the proposed Gauss–Newton type iteration linearly converges to the local solution (see Teunissen 1990).

Numerical examples

The main purpose of this section is to illustrate the proposed ICTLS algorithm through geodetic applications. In this first example, we use the data presented in Peng *et al.* to compute the ICTLS results. Table 1 gives the coefficient matrix $\mathbf{A}$ and the observation vector $\mathbf{y}$, as well as values for the inequality $\mathbf{B}_0\boldsymbol{\xi} \leq \mathbf{d}_0$ and the box constraints

Table 1 Data from Peng *et al.* and Zhang *et al.* (2013)

		A		y
0.9501	0.7620	0.6153	0.4057	0.0578
0.2311	0.4564	0.7919	0.9354	0.3528
0.6068	0.0185	0.9218	0.9169	0.8131
0.4859	0.8214	0.7382	0.4102	0.0098
0.8912	0.4447	0.1762	0.8936	0.1388
		$\mathbf{B}_0$		$\mathbf{d}_0$
0.2027	0.2721	0.7467	0.4659	0.5251
0.1987	0.1988	0.4450	0.4186	0.2026
0.6037	0.0152	0.9318	0.8462	0.6721
Box constraints: $-0.1 \le \xi_i \le 2.0$, $i = 1, 2, 3, 4$				

$-0.1 \le \xi_i \le 2.0$, $i = 1, 2, 3, 4$. The inequality constraints, relating only to the parameter vector, can be formulated together as

Inequality constraints for the parameter vector

$$\left[-\mathbf{B}_0^T \quad \mathbf{I}_4 \otimes \left[1 \quad -1\right]\right]^T \boldsymbol{\xi} \ge \left[-\mathbf{d}_0^T \quad \mathbf{1}_4 \otimes \left[-0.1 \quad -2\right]\right]^T$$

After computing the TLS solution with inequality constraints for the parameter vector, we present the results in Table 2. The results of ICTLS in the second column of Table 2 correspond exactly to the results presented in Zhang *et al.* (2013); however, our method does not use combinatorial strategies that can lead to large computational expenses. As the next step, we artificially add some constraints for the independent variables as follows:

Inequality constraints for the dependent variables

$$0.9 \le \bar{a}_{11} \le 1$$
$$0.7 \le \bar{a}_{12} \le 0.8$$
$$0.6 \le \bar{a}_{13} \le 0.7$$
$$0.4 \le \bar{a}_{14} \le 0.5$$

These correspond to the first row in the true design matrix being restricted without loss of generality.

The results in the third column of Table 2 present our estimates, which fulfil the inequality constraints both for the independent variables and the parameter vector. The parameter estimates do not significantly differ from the ICTLS results in the second column. The small difference may be explained by the verification that only the last constraint for the independent variables is active. Due to the additional inequality constraints for dependent variables, the TSSR is larger than that in the second column.

For the next test, we add the constraints for the dependent variables:

Inequality constraints for the dependent variables

$$0.3 \le \bar{y}_1 \le 0.5$$
$$0 \le \bar{y}_4 \le 0.1$$

These correspond to the first and the fourth elements within the observation vector.

The ICTLS results for restricting the parameter vector, independent variables and dependent variables simultaneously are presented in the fourth column of Table 2. The parameter estimates significantly differ from the ICTLS results presented in the second and third columns. The difference can be explained by the fact that additional constraints for the dependent variables may change the active sets of the inequality constraints for the independent variables and the parameter vector. As expected, the TSSR in this case is significantly larger than the first two tests.

Table 2 The results of the ICTLS

	ICTLS (only for the parameter vector)	ICTLS (for the parameter vector and dependent variables)	ICTLS (for the parameter vector, independent variables and dependent variables)
$\hat{\xi}_1$	−0.100000	−0.099998	0.087190
$\hat{\xi}_2$	−0.100000	−0.099999	−0.100000
$\hat{\xi}_3$	0.168547	0.167939	0.472197
$\hat{\xi}_4$	0.399777	0.400421	−0.011879
TSSR	0.139737	0.139786	0.222367

Similarity transformation with inequality constraints for dependent variables

In the second example, a 2D similarity transformation problem is presented to test the proposed algorithms. The functional model of the similarity transformation in 2-D space is considered as follows

$$\begin{bmatrix} X \\ Y \end{bmatrix} \approx s \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \qquad (11)$$

where $\left[x \quad y\right]^T$ and $\left[X \quad Y\right]^T$ are observed coordinates of the source system and the target system, respectively, α is the rotation angle, s is the scale and $\left[\Delta x \quad \Delta y\right]^T$ are the translations in the x and y orientations.

Therefore, the whole equation system can be written as follows:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} \approx \begin{bmatrix} x_1 & -y_1 & 1 & 0 \\ x_2 & -y_2 & 1 & 0 \\ x_3 & -y_3 & 1 & 0 \\ x_4 & -y_4 & 1 & 0 \\ y_1 & x_1 & 0 & 1 \\ y_2 & x_2 & 0 & 1 \\ y_3 & x_3 & 0 & 1 \\ y_4 & x_4 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \qquad (12)$$

where $a = s\cos\alpha$, $b = -s\sin\alpha$, $c = \Delta x$ and $d = \Delta y$.

The data for the 2-D similarity transformation were provided in Fang (2013). They are comprised of 2-D coordinates of four points from both the source and target systems listed in Table 3 (unit: metre). It is noted that the source and target data are i.i.d.

Table 3 Coordinate estimates in source and target systems

Point number	X (target)	Y (target)	x (source)	y (source)
1	−117.478	0	17.856	144.794
2	117.472	0	252.637	154.448
3	0.015	−117. 41	140.089	32.326
4	−0.014	117.451	130.40	267.027

Table 4 Parameter estimates of the transformation parameters

Parameters	WTLS algorithm without constraints	WTLS with inequality constraints for dependent variables
$\hat{a}$	0.999007	0.999807
$\hat{b}$	−0.041098	−0.041245
$\hat{c}$	−141.262790	−141.270777
$\hat{d}$	−143.931643	−143.938367

By incorporating prior information (the information from the previous adjustment), we have to reduce the uncertainty of the source coordinates, which means that the target coordinates are required in the given interval round the observations. In this example, we apply the interval ± 1 mm with the centre at the vector of the target coordinates, which means that all adjusted observations of the target coordinates are required within the interval.

We implement our algorithm (WTLS with inequality constraints for dependent variables) by adapting the above-mentioned prior information, and show the results in Table 4. The translates differ from the WTLS results up to the cm level whereas the product of the scale factor and the rotation sine and cosine hold at the digit 10^{-3}. Therefore, the prior information represented by the inequality constraints for the dependent variables is properly incorporated in the WTLS adjustment, and practically reduce the uncertainty of the target observations which was controlled by the previous adjustment.

Conclusion and outlook

In this work, we have reviewed all the reformulations of the EIV model in terms of the conventional adjustment models, and described algorithms to solve TLS problems with inequality constraints that restrict the parameter vector, independent variables and dependent variables. The proposed ICTLS algorithm is based on one reformulation of the EIV model, the model with nonlinear constraints. In the process of implementing the ICTLS algorithm, quadratic programming is solved iteratively, with the parameter vector, true design matrix and true observation vector all treated as unknowns. Our algorithm successfully address the problem of the similarity transformation with the prior information of the target observations represented by inequality constraints. As further study, the statistical analysis including the mean squared error for the ICTLS solution should be investigated. Since Amiri-Simkooei *et al*. (2016b) proposed three strategies to compute the covariance matrix of unconstrained estimate, we will generalise the approach to the inequality case.

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